Contents

1 Introduction

This essay delves into the realm of Quantum Field Theory (QFT) and its fascinating extension into curved space-time, a concept that unveils unexpected phenomena like Hawking radiation. Primarily drawing from the lecture notes of L.H. Ford [2] and Stephen Hawking's orginal article [3], this text aims to illuminate the intriguing process of extending QFT to the context of a Schwarzschild black hole, characterized by the absence of angular momentum and electric charge. This approach provides a clear example of how black holes emit particles with a thermal spectrum, offering insights into the complexities of QFT in curved space-time.

The initial section of the essay presents an overview of Quantum Field Theory in the context of flat space-time, followed by an examination of its application in curved space-time surrounding black holes. Subsequently, the essay delves into demonstrating Hawking radiation, specifically through the simplified case of massless, scalar fields within Schwarzschild space-time. The discussion aims to unpack the process behind Hawking radiation, focusing on both its derivation and broader implications, including the black hole information paradox.

2 Quantum Field Theory in Curved Space-Time

Quantum Field Theory (QFT) links together the ideas of special relativity, quantum mechanics, and classical field theory into a coherent framework. It respects Lorentz invariance, according to relativity, and adopts field quantization in harmony with quantum mechanics. Using the Lagrangian method it allows to study the dynamics and interactions of fields. In the flat, Minkowski space-time, particles are then defined as quantized excitations of their underlying quantum fields, representing discrete units of energy and momentum. In this context QFT has offered precise predictions for a variety of applications, including the standard model of particle physics.

However, the universe doesn't always present a flat space-time. General Relativity describes space-time as a four-dimensional surface that can be distorted by the gravitational pull of massive entities like black holes. This curvature necessitates a rethinking of QFT's fundamental principles. Adapting QFT to curved space-time presents challenges, especially in defining core concepts such as particles, therefore adding layers of complexity to the theory's interpretation within this new framework. This section will explore these challenges.

Drawing inspiration from Stephen Hawking's original work [3] and L.H. Ford's lecture notes [2], this approach treats space-time metric classically while applying quantum mechanics to the interaction with the matter field. Matter fields obey standard wave equations, with a critical difference: the Minkowski metric $\eta_{\mu\nu}$ is replaced by the Schwarzschild metric $g_{\mu\nu}$:

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \rightarrow \quad g_{\mu\nu} = \begin{bmatrix} \frac{r-2M}{r} & 0 & 0 & 0 \\ 0 & -\frac{r-2M}{r}^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix} \tag{1}$$

This metric encapsulates space-time under the influence of a massive, non-rotating, spherically symmetric object, such as an uncharged, non-rotating black hole. The invariant square of an infinitesimal line element then becomes :

$$ds_\eta^2 = dt^2 - dr^2 - r^2 d\Omega^2 \quad \rightarrow \quad ds_g^2 = (\frac{r-2M}{r})dt^2 - (\frac{r-2M}{r})^{-1} dr^2 - r^2 d\Omega^2 \tag{2}$$

Where natural units are used ($G = c = \hbar = 1$), $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and M is the black hole's mass.

The discussion will center on a real, massive scalar field denoted by ϕ to simplify discussions. While the classical Klein-Gordon equation in flat space-time typically yields a general solution in the form of a real superposition of plane waves [4], employing the Schwarzschild metric leads to a significantly more complex equation, which in turn results in more complex solutions [5]:

$$\begin{aligned}(\nabla_\mu \nabla^\mu + m^2)\phi &= 0 \\ &= g^{tt}\partial_t^2\phi + \frac{1}{\sqrt{|g|}}\partial_{r_*}[\sqrt{|g|}g^{r_* r_*}\partial_{r_*}\phi] + \frac{g^{\theta\theta}}{\sqrt{|g|}}\partial_\theta[\sqrt{|g|}\partial_\theta\phi] + g^{\varphi\varphi}\partial_\varphi^2\phi + m^2\phi\end{aligned} \tag{3}$$

With the square root of the metric determinant $\sqrt{|g|} = \frac{r-2M}{r} r^2 \sin\theta$ and the tortoise coordinate $r_* = r + 2Mln(r/2M - 1)$. The field can then be expanded in modes $f_{\omega lm}(r,t,\theta,\varphi) = r^{-1} f_{\omega l}(r) Y_{lm}(\theta,\phi) e^{-i\omega t}$ where Y_{lm} are the spherical harmonics and $f_{\omega l}(r)$ satisfies [1] :

$$\frac{\partial^2 f_{\omega l}}{\partial r_*^2} + (\ \omega^2 - [\ m^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3}\ ]\ \times\ [\ 1 - \frac{2M}{r}\ ]\ ) f_{\omega l} = 0 \tag{4}$$

Having demonstrated a method to transition from Quantum Field Theory in flat space-time to a Schwarzschild space-time, the next step is to explore how quantizing a field in curved space-time introduces difficulties in defining a unique vacuum state. This issue further complicates the interpretation of what constitutes a particle in such contexts.

Canonical quantization elevates the field $\phi(\mathbf{x}, t)$ and its conjugate momentum $\Pi(\mathbf{x}', t)$ to operator status, adhering to equal time commutation relations:

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{x}', t)] = \delta(x, x') \tag{5}$$

Using the index i for abstract purposes in order to facilitate discussions, the field expansion can be expressed as a sum of annihilation and creation operators :

$$\phi = \sum_i (f_i a_i + f_i^* a_i^\dagger) \tag{6}$$

In flat space-time, $\{f_i\}$ and $\{f_i^*\}$ symbolize complete sets of solutions with positive and negative frequencies, respectively, relative to the conventional Minkowski time coordinate. Thus, $\{f_i, f_i^*\}$ form a complete, orthonormal, and unique set of solutions to the wave equation, allowing the interpretation of a_i and $a_i^\dagger$ as particle annihilation and creation operators for the i^{th} state. The vacuum state $|0\rangle$ is defined as the unique state from which no further particle annihilation is possible, such that $a_i |0\rangle = 0$ for all i.

However, in curved space-time, gravitational field-induced space-time curvature affects time's flow, varying with gravitational strength. This variance prevents the establishment of a universal standard of time and thus the division into positive and negative frequencies becomes not feasible. Although it is still possible to define sets $\{f_i\}$ and $\{f_i^*\}$ as complete and orthonormal solutions:

$$\begin{aligned}(f_i, f_j) = i\int f_j^* \overleftrightarrow{\partial}_\mu f_i d\Sigma^\mu &= \delta_{i,j} \\ (f_i, f_j^*) = i\int f_j \overleftrightarrow{\partial}_\mu f_i d\Sigma^\mu &= 0\end{aligned} \tag{7}$$

the conditions are not sufficient for a unique choice of $\{f_i\}$ and $\{f_i^*\}$.

A particularly interesting scenario is asymptotically flat space-time, which can be divided into three regions: initially flat space-time in the past *(1^{st} region)*, a non-flat intermediate region *(2^{nd} region)*, and flat space-time in the future *(3^{rd} region)*. In the past and future, sets $\{f_{1i}\}$ and $\{f_{3i}\}$ can respectively be defined, containing only positive frequencies with respect to the Minkowski time coordinate in the 1^{st} and 3^{rd} regions. These sets also adhere to orthogonality relations:

$$(f_{1i}, f_{1j}) = (f_{3i}, f_{3j}) = \delta_{i,j} \tag{8}$$

$$(f_{1i}, f_{1j}^*) = (f_{3i}, f_{3j}^*) = 0 \tag{9}$$

Moreover, the sets $\{f_i\}$ and $\{f_{3i}\}$ being different solutions of the wave equation everywhere in space-time, the field operator can be expanded according to either of these sets:

$$\phi = \sum_i (a_{1i} f_{1i} + a_{1i}^\dagger f_{1i}^*) = \sum_i (a_{3i} f_{3i} + a_{3i}^\dagger f_{3i}^*) \tag{10}$$

The annihilation and creation operators in the past and future (a_{1i}, $a_{1i}^\dagger$, a_{3i}, $a_{3i}^\dagger$) demonstrate that an initial vacuum state in the past ($|0_1\rangle$) does not equate to a vacuum state in the future ($a_{3i}|0_1\rangle \neq 0$), leading to an ambiguity in particle definition. This can be seen as if the fluctuating metric (or gravitational field) was generating a specific quantity of particles associated with the scalar field. Further exploration reveals relationships between past and future modes:

$$f_{1i} = \sum_k (\alpha_{ik} f_{3k} + \beta_{ik} f_{3k}^*) \tag{11}$$

$$f_{3k} = \sum_i (\alpha_{ik}^* f_{1i} - \beta_{ik} f_{1i}^*) \tag{12}$$

Given that $a_{1i} = (\phi, f_{1i})$ and $a_{3i} = (\phi, f_{3i})$, the Bogolubov transformations, expressing relations between two sets of operators, can be derived:

$$a_{1i} = \sum_k (\alpha^*_{ik} a_{3k} - \beta^*_{ik} a^\dagger_{3k}) \qquad (13) \qquad\qquad a_{3i} = \sum_i (\alpha_{ik} a_{1i} + \beta^*_{ik} a^\dagger_{1i}) \qquad (14)$$

Inserting (11) into the orthogonality relations (8) and (9), one can derive conditions on the Bogolubov coefficients :

$$\sum_k (\alpha_{ik} \alpha j k^* - \beta_{ik} \beta j k^*) = \delta_{ij} \qquad (15)$$

This relationship will be especially useful in the upcoming section for calculating the average number of particles emitted by the black.

3 Black Holes and Hawking Radiation

Hawking radiation emerges as the thermal black body radiation from black holes. Although yet to be observed experimentally due to its low intensity, far below the sensitivity of current telescopes, it stands as a profound example of quantum field theory in curved space-time.

Consider, for illustrative simplicity, the case of a Schwarzschild black hole. Such black holes theoretically form from the gravitational collapse of non-spinning objects, resulting in entities devoid of angular momentum and electric charge. The Penrose diagram of a Schwarzschild black hole is illustrated in Figure 1(a), offering a comprehensive yet compact visualization of spacetime's causal structure. This diagram effectively distills the vastness of spacetime into a finite representation. In the diagram, $\mathcal{I}^-$ and $\mathcal{I}^+$ represent the concepts of the infinite past and future. The area shaded in the diagram indicates the interior region of the collapsing body. Notably, the trajectory of a light ray in this depiction is represented by a line at a 45° angle. Consequently, the event horizon, also inclined at 45°, dictates that any object crossing it inexorably propagates towards the black hole's center. The singularity, a point of zero volume and infinite density, signifies the end of space and time.

The objective here is to demonstrate the thermal spectrum of the particles produced by the black hole. Following the derivation in [2], inspired by Stephen Hawking's original work [3], the focus is drawn on massless, scalar fields in Schwarzschild spacetime. However as shown in Hawking's article, those results are applicable to any quantum field in general black hole spacetimes. In the described approach, the space-time is divided into three regions, with $\mathcal{I}^-$ and $\mathcal{I}^+$ representing the so-called past and future, the middle region corresponding to the black hole space. Concentrating on particle creation at late times where modes with high frequencies are the most significant, this allows to approximate their propagation through geometrical optics. As illustrated in figure 1(b), for $v > v_0$, a light ray traverses the black hole at constant $v = g^{-1}(u)$, emerging at constant $u = g(v)$. For $v < v_0$, the ray cannot escape and is doomed to reach the singularity.

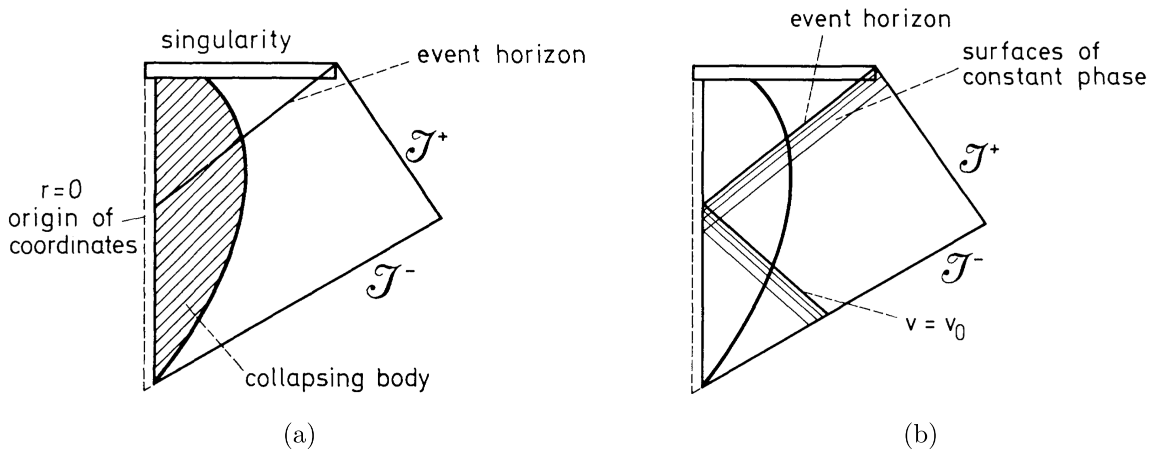

Figure 1: Penrose diagran of a schwarzschild black hole. taken form [3].

Defining the retarded and advanced time coordinates as $v = t + r_*$ and $u = t - r_*$, and expanding the massless field in modes $f_{\omega lm}$ as per eq. (3), the past modes align with positive frequency on $\mathcal{I}^-$, and future modes correspond to positive frequency on $\mathcal{I}^+$ such that the asymptotic solutions of the wave equation for massless scalar fields yields the following modes :

$$f_{1,\omega lm} \sim \frac{Y_{lm}}{\sqrt{4\pi\omega r}} \times \begin{cases} e^{-i\omega v} \text{on } \mathcal{I}^- \\ e^{-i\omega g^{-1}(u)} \text{on } \mathcal{I}^+ \end{cases} \qquad f_{3,\omega lm} \sim \frac{Y_{lm}}{\sqrt{4\pi\omega r}} \times \begin{cases} e^{-i\omega g(v)} \text{on } \mathcal{I}^- \\ e^{-i\omega u} \text{on } \mathcal{I}^+ \end{cases}$$

The case of a thin shell, as elaborated in [2], provides a clear resolution, though the results generalize as shown in [3]. Within this framework, the space inside the shell is flat, whereas the space outside is described by Schwarzschild spacetime. The null coordinates are then defined differently in these two regions: inside the shell, they are denoted as $V = T + r$ and $U = T - r$, and outside, as $u = t + r_*$ and $v = t - r_*$. By aligning the metrics at the shell's boundary, a specific condition arises, serving as a crucial junction between these two areas :

$$1 - (\frac{dr}{dT})^2 = (\frac{r - 2M}{r})(\frac{dt}{dT})^2 - (\frac{r - 2M}{r})^{-1}(\frac{dr}{dT})^2 \tag{16}$$

Deriving conditions at the shell's entry, center, and exit, and combining these yields the relationship between v and u:

$$u = g(v) = -4Mln(\frac{v_0 - v}{C}) \tag{17}$$

With C as a constant. The relation between Bogolubov coefficients can then be determined by taking the Fourier transform of $f_{3,\omega lm}$:

$$f_{3,\omega lm} = \int_0^\infty d\omega' (\alpha^*_{\omega' \omega lm} f_{1,\omega lm} - \beta_{\omega' \omega lm} f^*_{1,\omega lm}) \tag{18}$$

This leads to the expressions for the coefficients :

$$\begin{aligned} \alpha^*_{\omega' \omega lm} &= \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{v_0}^\infty dv e^{i\omega' v} e^{4Mi\omega ln[(v_0 - v)/C]} \\ \beta_{\omega' \omega lm} &= -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{v_0}^\infty dv e^{-i\omega' v} e^{4Mi\omega ln[(v_0 - v)/C]} \end{aligned} \tag{19}$$

Complex analysis of those two expressions allows to express the relationship between those coefficients as :

$$|\alpha_{\omega' \omega lm}| = e^{4\pi M\omega} |\beta_{\omega' \omega lm}| \tag{20}$$

Applying the condition (15) to this relation finnaly yields :

$$\sum_{\omega'} (|\alpha_{\omega' \omega lm}|^2 - |\beta_{\omega' \omega lm}|^2) = \sum_{\omega'} (e^{8\pi M\omega} - 1)|\beta_{\omega' \omega lm}|^2 = 1 \tag{21}$$

Allowing to calculate the mean number of particles created in the mode ωlm :

$$N_{\omega lm} = \sum_{\omega'} |\beta_{\omega' \omega lm}|^2 = \frac{1}{e^{8\pi M\omega} - 1} \tag{22}$$

This leads to the identification of a Planck spectrum, characteristic of black body radiation, with a black hole's temperature defined as $T = \frac{1}{8\pi M}$. This process of thermal emission should then gradually reduces the mass of the black hole, which in turn leads to a decrease in the area of the black hole's event horizon. However, in classical physics, a black hole's event horizon is thought to never decrease because nothing should be able to emerge from it. Therefore, the outcome derived in (22) sands strongly in contrast with what classical physics would anticipate. Stephen Hawking offered a theoretical explanation for this phenomenon.

He suggested that this apparent violation occurs due to a flux of negative energy entering the black hole. Typically, at the event horizon, virtual particle-antiparticle pairs are created and then annihilate each other. However, in the context of Hawking radiation, these pairs can become separated. The antiparticle can tunnel into the black hole, where it becomes a real particle, effectively reducing the black hole's mass. Meanwhile, the particle with positive energy can escape to infinity, contributing to the black hole's thermal radiation. This process, therefore, provides a mechanism by which a black hole can gradually lose mass and ultimately shrink in size. But note that Hawking itself prevented to take its interpretation too literally as it lacks a more complete quantum theory of gravity.

Moreover, those results give rise to other various intriguing questions, notably the black hole information paradox. Indeed, this thermal nature of black holes suggests a potential loss of information, due to the absence of correlations in the emitted particles. The information within a black hole seems to disappear as it dissipates. There have been attempts to resolve this paradox, including theories suggesting that perturbations in Hawking radiation might carry the missing correlations in the outgoing radiation. This perspective postulates that information might be released gradually during the black hole's evaporation process.

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